

Johnson-Segalman – Saint-Venant equations for viscoelastic shallow flows in the elastic limit

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Abstract The shallow-water equations of Saint-Venant, often used to model the long-wave dynamics of free-surface flows driven by inertia and hydrostatic pressure, can be generalized to account for the elongational rheology of non-Newtonian fluids too. We consider here the 4×4 shallow-water equations generalized to *viscoelastic* fluids using the Johnson-Segalman model in the elastic limit (i.e. at infinitely-large Deborah number, when source terms vanish). The system of nonlinear first-order equations is hyperbolic when the *slip parameter* is small $\zeta \leq \frac{1}{2}$ ($\zeta = 1$ is the corotational case and $\zeta = 0$ the upper-convected Maxwell case). Moreover, it is naturally endowed with a mathematical entropy (a physical free-energy). When $\zeta \leq \frac{1}{2}$ and for any initial data excluding vacuum, we construct here, when elasticity $G > 0$ is non-zero, the unique solution to the Riemann problem under Lax admissibility conditions. The standard Saint-Venant case is recovered when $G \rightarrow 0$ for small data.

1 Setting of the problem

The well-known one-dimensional shallow-water equations of Saint-Venant

$$\partial_t h + \partial_x(hu) = 0 \quad (1)$$

$$\partial_t(hu) + \partial_x(hu^2 + gh^2/2) = 0 \quad (2)$$

model the dynamics of the mean depth $h(t, x) > 0$ of a perfect fluid flowing with mean velocity $u(t, x)$ on a flat open channel with uniform cross section along a straight axis \mathbf{e}_x , under gravity (perpendicular to \mathbf{e}_x , with constant magnitude g).

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Now, following the interpretation of (1–2) as an approximation of the depth-averaged Free-Surface Navier-Stokes (FSNS) equations governing Newtonian fluids, and starting depth-averaging from the FSNS/Upper-Convected-Maxwell(UCM) system of equations for (linear) *viscoelastic* fluids [3, 4], one can in fact derive a *generalized Saint-Venant* (gSV) system of shallow-water-type equations

$$\partial_t h + \partial_x(hu) = 0 \quad (3)$$

$$\partial_t(hu) + \partial_x(hu^2 + gh^2/2 + hN) = 0 \quad (4)$$

where the normal stress difference term in the momentum balance $N = \tau_{zz} - \tau_{xx}$ is function of additional extra-stress variables $\tau_{zz}(t, x)$, $\tau_{xx}(t, x)$ governed by e.g.

$$\tau_{xx} + \lambda(\partial_t \tau_{xx} + u\partial_x \tau_{xx} - 2\tau_{xx}\partial_x u) = \nu \partial_x u \quad (5)$$

$$\tau_{zz} + \lambda(\partial_t \tau_{zz} + u\partial_x \tau_{zz} + 2\tau_{zz}\partial_x u) = -\nu \partial_x u \quad (6)$$

i.e. depth-averaged UCM equations modelling *elongational* viscoelastic effects.

When the relaxation time is small $\lambda \rightarrow 0$ (i.e. the Deborah number, when $\lambda > 0$ is non-dimensionalized with respect to a time scale characteristic of the flow) the system (3–4–5–6) converges to standard viscous Saint-Venant equations with viscosity $\nu \geq 0$. When the relaxation time *and the viscosity* are equivalently large $\lambda \sim \nu \rightarrow +\infty$, the system (3–6) converges to elastic Saint-Venant equations (in Eulerian formulation, see e.g. [8]) with elasticity $G = \nu/(2\lambda) \geq 0$, which coincides with the homogeneous version of the system (7–8) (i.e. when the source term vanish)

$$\partial_t \sigma_{xx} + u\partial_x \sigma_{xx} - 2\sigma_{xx}\partial_x u = (1 - \sigma_{xx})/\lambda \quad (7)$$

$$\partial_t \sigma_{zz} + u\partial_x \sigma_{zz} + 2\sigma_{zz}\partial_x u = (1 - \sigma_{zz})/\lambda \quad (8)$$

obtained after rewriting (5–6) using $N = G(\sigma_{zz} - \sigma_{xx})$, and $\tau_{xx,zz} = G(\sigma_{xx,zz} - 1)$.

More general evolution equations of differential rate-type for the extra-stress, the Johnson-Segalman (JS) equations with slip parameter $\zeta \in [0, 2]$, can also be coupled to FSNS before depth-averaging. In fact (7–8) arise in the specific case $\zeta = 0$ (upper-convected Gordon-Schowalter derivative) for gSV system (3–4–9–10)

$$\partial_t \sigma_{xx} + u\partial_x \sigma_{xx} + 2(\zeta - 1)\sigma_{xx}\partial_x u = (1 - \sigma_{xx})/\lambda \quad (9)$$

$$\partial_t \sigma_{zz} + u\partial_x \sigma_{zz} + 2(1 - \zeta)\sigma_{zz}\partial_x u = (1 - \sigma_{zz})/\lambda \quad (10)$$

that accounts for *linear* viscoelastic elongational effects standardly established for e.g. polymeric liquids [1]. The gSV system with JS is already an interesting starting point for mathematical studies, although it could still be further complicated to account for more established physics ; we refer to [1] for details.

In the following, we consider the Cauchy problem on $t \geq 0$ for the quasilinear gSV system (3–4–9–10) when it is supplied by an initial condition with bounded total variation. Weak solutions with bounded variations (BV) on $\mathbb{R} \ni x$ can be constructed for quasilinear systems provided the system is *strictly hyperbolic*, in particular when characteristic fields are genuinely nonlinear or linearly degenerate [5, 7].

First, we show that gSV is hyperbolic for all $h \geq 0, \sigma_{xx} > 0, \sigma_{zz} > 0$ provided $\zeta \leq \frac{1}{2}$; *strictly* provided $h > 0$, or $G > 0$ and, $\zeta > 0$ or $\sigma_{xx} \neq \sigma_{zz}$. Next, assuming $\zeta \leq \frac{1}{2}$ and $h > 0$, we construct univoque gSV solutions guided by the dissipation rule

$$\partial_t F + \partial_x (u(F + P)) \leq G(2 - \sigma_{xx} - \sigma_{xx}^{-1} + 2 - \sigma_{zz} - \sigma_{zz}^{-1}) / (2\lambda) \quad (11)$$

for the same mathematical entropy F as for $\zeta = 0$ [3] as admissibility criterion

$$F = h(u^2 + gh + G(\sigma_{xx} + \sigma_{zz} - \ln \sigma_{xx} - \ln \sigma_{zz} - 2)) / 2, \quad (12)$$

denoting $P = gh^2/2 + hN$. Smooth gSV solutions obviously satisfy the *equality* (11). When $h, \sigma_{xx}, \sigma_{zz} > 0$, gSV reads as a system of conservation laws rewriting (9–10)

$$\partial_t (h \log(h^{2(1-\xi)} \sigma_{xx})) + \partial_x (hu \log(h^{2(1-\xi)} \sigma_{xx})) = h(\sigma_{xx}^{-1} - 1)/\lambda, \quad (13)$$

$$\partial_t (h \log(h^{2(\xi-1)} \sigma_{zz})) + \partial_x (hu \log(h^{2(\xi-1)} \sigma_{zz})) = h(\sigma_{zz}^{-1} - 1)/\lambda. \quad (14)$$

But whereas F is convex in e.g. $(h, hu, h\sigma_{xx}, h\sigma_{zz})$, see [3] when $\xi = 0$, it cannot be convex with respect to any variable $V = (h, hu, h\mathcal{X}(\sigma_{xx}h^{2(1-\xi)}), h\mathcal{Z}(\sigma_{zz}h^{2(\xi-1)}))$ using smooth $\mathcal{X}, \mathcal{Z} \in C^1(\mathbb{R}_*^+, \mathbb{R}_*^+)$ such that the system rewrites

$$\partial_t V + \partial_x \mathbb{F}(V) = 0, \quad (15)$$

$\mathbb{F}(V) = (hu, hu^2 + gh^2/2 + Gh\sigma_{zz} - Gh\sigma_{xx}, hu\mathcal{X}(\sigma_{xx}h^{2(1-\xi)}), hu\mathcal{Z}(\sigma_{zz}h^{2(\xi-1)}))$. Now, whereas *univoque* solutions to quasilinear (possibly non-conservative) systems can be constructed using (convex) entropies [7], conservative formulations alone (without admissibility criterion) are not enough. This is why we carefully investigate the building-block of univoque BV solutions: univoque solutions to Riemann initial-value problems for a quasilinear system (16) in well-chosen variable U

$$\partial_t U + A(U)\partial_x U = S(U) \quad (16)$$

in the homogeneous case $S \equiv 0$ (obtained in the limit $\lambda \rightarrow \infty$). Precisely, when $\zeta \leq \frac{1}{2}$ and $G > 0$ we build the unique weak solutions $U(t, x)$ *admissible under Lax condition* to Riemann problems for (16) with piecewise-constant initial conditions

$$U(t \rightarrow 0^+, x) = \begin{cases} U_l & x < 0 \\ U_r & x > 0 \end{cases} \quad (17)$$

given *any* states $U_l, U_r \in \mathcal{U}$ in the strict hyperbolicity region $\mathcal{U} = \{h > 0, \sigma_{xx} > 0, \sigma_{zz} > 0\} \subset \mathbb{R}^d$. Our Riemann solutions satisfy the conservative system (15) in the distributional sense on $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$ and are consistent with the standard Saint-Venant case when $G \rightarrow 0$. These Riemann solutions are a key tool to define weak BV solutions to the Cauchy problem for gSV which are unique within the admissible BV solutions' class modulo some restriction on oscillations, see e.g. [7, Chap.X].

However, note that the vacuum state $h = 0$ shall never be reached as a limit state by any sequence of admissible Riemann solutions when $G > 0$, as opposed to the standard Saint-Venant case $G = 0$ (like in the famous Ritter problem for instance). This is in fact related to well-posedness in the large (i.e. for any initial condition $U_l, U_r \in \mathcal{U}$) when $G > 0$, as opposed to the standard Saint-Venant case $G = 0$ (so the latter case is some kind of singular limit): when $G > 0$, gSV impulse blows up as $h \rightarrow 0$, so vacuum cannot be reached, while Hugoniot curves in turn span the whole range \mathcal{U} . Also, consistently with the occurrence of vacuum when $G = 0$ (standard Saint-Venant), the latter case can be recovered when $G \rightarrow 0$ only for *small* initial data (otherwise, the intermediate state in Riemann solution blows up).

2 Hyperbolic structure of the system of equations

Given $g > 0, G \geq 0$, consider first the gSV system (3–4–9–10) written in the non-conservative quasilinear form (16) using the variable $U = (h, u, \sigma_{xx}, \sigma_{zz}) \in \mathcal{U}$. One easily sees that $\lambda^0 := u$ is an eigenvalue with multiplicity two for the jacobian matrix A associated with the linearly degenerate 0-characteristic field (i.e. $r^0 \cdot \nabla_U \lambda^0 = 0$)

$$r^0 \in \text{Span}\{r_1^0, r_2^0\} \quad r_1^0 := \begin{pmatrix} Gh \\ 0 \\ (gh + N) \\ 0 \end{pmatrix} \quad r_2^0 := \begin{pmatrix} Gh \\ 0 \\ 0 \\ -(gh + N) \end{pmatrix} \quad (18)$$

with Riemann invariants u, P (i.e. $r^0 \cdot \nabla_U P = 0$). Moreover, as long as $\zeta \leq \frac{1}{2}$, holds

$$\partial_h P|_{\sigma_{xx}h^{2(1-\zeta)}, \sigma_{zz}h^{2(\zeta-1)}} = gh + G(\sigma_{zz} - \sigma_{xx}) + 2G(1 - \zeta)(\sigma_{zz} + \sigma_{xx}) > 0 \quad (19)$$

for $h, \sigma_{xx}, \sigma_{zz} > 0$ so, after computations, the two other eigenvalues of A are real

$$\lambda^\pm := u \pm \sqrt{\partial_h P|_{\sigma_{xx}h^{2(1-\zeta)}, \sigma_{zz}h^{2(\zeta-1)}}} \quad (20)$$

and define two genuinely nonlinear fields (denoted by $+$ and $-$) spanned by

$$r^\pm := \begin{pmatrix} h \\ \pm \sqrt{\partial_h P|_{\sigma_{xx}h^{2(1-\zeta)}, \sigma_{zz}h^{2(\zeta-1)}}} \\ 2(\zeta - 1)\sigma_{xx} \\ 2(1 - \zeta)\sigma_{zz} \end{pmatrix} \quad (21)$$

with $\sigma_{xx}h^{2(1-\zeta)}, \sigma_{zz}h^{2(\zeta-1)}$ as Riemann invariants ; note in particular for $\zeta \in [0, \frac{1}{2}]$

$$r^\pm \cdot \nabla_U \lambda^\pm = \pm \frac{3gh + 2G(3 - 2\zeta)(2 - \zeta)\sigma_{zz} + 2G\zeta(1 - 2\zeta)\sigma_{xx}}{2\sqrt{\partial_h P|_{\sigma_{xx}h^{2(1-\zeta)}, \sigma_{zz}h^{2(\zeta-1)}}}} \gtrless 0. \quad (22)$$

Univoque piecewise-smooth solutions to Cauchy-Riemann problems for (3–4–9–10) read $U(t, x) = \tilde{U}(x/t)$ with $\tilde{U}(\xi)$ piecewise differentiable solution on $\mathbb{R} \ni \xi$ to

$$\xi \tilde{U}' = A(\tilde{U}) \tilde{U}' \quad \tilde{U} \xrightarrow[\xi \rightarrow -\infty]{} U_l, \quad \tilde{U} \xrightarrow[\xi \rightarrow +\infty]{} U_r \quad (23)$$

having finitely-many discontinuities ξ_m , $m = 0 \dots M$, shall next be constructed for any initial condition $U_l, U_r \in \mathcal{U}$ using elementary waves satisfying $\tilde{U}' \in \text{Span } r^\pm$, $\xi = \lambda^\pm$ therefore $\tilde{U}' = r^\pm / (r^\pm \cdot \nabla_U \lambda^\pm)$, or $\tilde{U}' = 0$, and an admissibility criterion.

3 Elementary-waves solutions

For all $U_l, U_r \in \mathcal{U}$, unique solutions to (16–17) shall be constructed in the form

$$\tilde{U}(\xi) = \begin{cases} U_l \equiv \tilde{U}_0 & \xi < \xi_0 \\ \tilde{U}_1(\xi) & \xi_0 < \xi < \xi_1 \\ \dots & \\ \tilde{U}_M(\xi) & \xi_{M-1} < \xi < \xi_M \\ U_r \equiv \tilde{U}_{M+1} & \xi_M < \xi \end{cases} \quad (24)$$

using M differentiable states \tilde{U}_m to connect $U_l, U_r \in \mathcal{U}$ through elementary waves.

3.1 Contact discontinuities and shocks

Elementary-waves solutions (24) with a single discontinuity ($M = 1$) shall be 0-contact discontinuities when, denoting Y_l (resp. Y_r) the left (resp. right) value of Y ,

$$\xi_0 = u_l = u_r \quad P_l = P_r \quad (25)$$

or \pm -shocks when, denoting $P_k = gh_k^2/2 + GZ^{-1}h_k^{1+2(1-\zeta)} - GXh_k^{1+2(\zeta-1)}$, hold

$$\xi_0(h_r - h_l) = (h_r u_r - h_l u_l), \quad (26)$$

$$\xi_0(h_r u_r - h_l u_l) = (h_r u_r^2 + P_r - h_l u_l^2 - P_l), \quad (27)$$

with 2 constants $Z^{-1} = \sigma_{zz,k} h_k^{2(\zeta-1)} > 0, X = \sigma_{xx,k} h_k^{2(1-\zeta)} > 0$ ($k \in \{l, r\}$), thus

$$u_r = u_l \pm \sqrt{(h_l^{-1} - h_r^{-1})(P_r - P_l)} \quad (28)$$

on combining (26), (27). Both waves satisfy Rankine-Hugoniot (RH) relationships

$$\xi_0(V_r - V_l) = \mathbb{F}_r - \mathbb{F}_l \quad (29)$$

and thus define standard weak solutions to (16) in the conservative variable $V(t, x) = \tilde{V}(x/t)$. Moreover, the entropy dissipation (11) in the elastic limit $\lambda \rightarrow \infty$

$$E := -\xi_0(F_r - F_l) + u(F + gh^2/2 + hN)|_r - u(F + gh^2/2 + hN)|_l \leq 0 \quad (30)$$

can be checked for contact discontinuities (as an equality), and for the *weak* shocks in the genuinely nonlinear fields λ^\pm (i.e. shocks with small enough amplitude) which satisfy Lax admissibility condition, see [6, 5] and [7, (1.24) Chap. VI]:

Lemma 1. *Right and left states V_r, V_l can be connected through an admissible*

- *--shock if $u_r = u_l - \sqrt{(h_l^{-1} - h_r^{-1})(P_r - P_l)}$, $h_r \geq h_l$*
- *+-shock if $u_r = u_l - \sqrt{(h_l^{-1} - h_r^{-1})(P_r - P_l)}$, $h_r \leq h_l$*

Indeed, it is enough that $F|_{X,Z}$ is convex in (h, hu) to discriminate against non-physical (weak) shocks, or equivalently, that $F|_{X,Z}/h$ is convex in (h^{-1}, u) [2].

Proof. $2F/h = u^2 + gh + G(\sigma_{xx} + \sigma_{zz} - \ln \sigma_{xx} - \ln \sigma_{zz} - 2)$ is convex in (h^{-1}, u) if

$$\partial_{h^{-2}}^2 \frac{F|_{X,Z}}{h} = \frac{2}{h^{-3}} \partial_h \frac{F|_{X,Z}}{h} + \frac{1}{h^{-4}} \partial_{h^2}^2 \frac{F|_{X,Z}}{h}$$

is positive, which holds when $\zeta \in [0, 1/2]$ such that

$$2\partial_h \frac{F|_{X,Z}}{h} = g + G(2(1 - \zeta)Zh^{2(1-\zeta)-1} + 2(\zeta - 1)Xh^{2(\zeta-1)-1}) \geq 0,$$

$$2\partial_{h^2}^2 \frac{F|_{X,Z}}{h} = G(2(1 - \zeta)(2(1 - \zeta) - 1)Zh^{-2\zeta} + 2(\zeta - 1)(2(\zeta - 1) - 1)Xh^{2(\zeta-2)}) \geq 0.$$

3.2 Rarefaction waves

Elementary waves with two discontinuities ($M = 2$) which are not a combination of two elementary waves with one discontinuity each shall be, on noting $k \in \{l, r\}$,

- a +-rarefaction wave if $h_l = h_0 < h_r = h_2$ such that for all $\xi \in (\xi_0 \equiv \lambda_l^+, \xi_2 \equiv \lambda_r^+)$

$$\begin{aligned} \xi &= \lambda_k^+ + \int_{h_k}^{h_1(\xi)} dh \frac{3gh + (4\zeta^2 - 14\zeta + 12)GZ^{-1}h^{2(1-\zeta)} + 2\zeta(1 - 2\zeta)GXh^{2(\zeta-1)}}{2h\sqrt{gh + (1 + 2(1 - \zeta))GZ^{-1}h^{2(1-\zeta)} - (1 + 2(\zeta - 1))GXh^{2(\zeta-1)}}} \\ u_1(\xi) &= u_k + \int_{h_k}^{h_1(\xi)} dh \sqrt{gh^{-1} + (1 + 2(1 - \zeta))GZ^{-1}h^{-2\zeta} - (1 + 2(\zeta - 1))GXh^{2(\zeta-2)}}, \end{aligned}$$

- a --rarefaction wave if $h_l = h_0 > h_r = h_2$ such that for all $\xi \in (\xi_0 \equiv \lambda_l^-, \xi_2 \equiv \lambda_r^-)$

$$\begin{aligned} \xi &= \lambda_k^- - \int_{h_k}^{h_1(\xi)} dh \frac{3gh + (4\zeta^2 - 14\zeta + 12)GZ^{-1}h^{2(1-\zeta)} + 2\zeta(1 - 2\zeta)GXh^{2(\zeta-1)}}{2h\sqrt{gh + (1 + 2(1 - \zeta))GZ^{-1}h^{2(1-\zeta)} - (1 + 2(\zeta - 1))GXh^{2(\zeta-1)}}} \\ u_1(\xi) &= u_k - \int_{h_k}^{h_1(\xi)} dh \sqrt{gh^{-1} + (1 + 2(1 - \zeta))GZ^{-1}h^{-2\zeta} - (1 + 2(\zeta - 1))GXh^{2(\zeta-2)}}. \end{aligned}$$

4 Solution to the general Riemann problem

The general Riemann problem can be solved by combining elementary waves [6]. Solutions (24) to systems with 3 characteristic fields require 3 backward characteristics through all points in $t > 0$, except on discontinuities, that are: $\xi_0 \leq \xi_1$ associated with the $-$ -field, $\xi_2 \in (\xi_1, \xi_3)$ associated with the 0-field, and $\xi_3 \leq \xi_4$ associated with the $+$ -field. So finally, a solution to the Riemann problem is characterized by

$$X_l = X_1 = X_2, Z_l = Z_1 = Z_2 \quad u_2 = u_3, P_2 = P_3 \quad X_r = X_4 = X_3, Z_r = Z_4 = Z_3 \quad (31)$$

with a (h_2, u_2) -locus given by $u_2 = u_l - \sqrt{(h_l^{-1} - h_2^{-1})(P_2 - P_l)}$ when $h_2 \geq h_l$ and

$$u_2 \leftarrow u_l(\xi) = u_l - \int_{h_l}^{h_1(\xi)} dh \sqrt{gh^{-1} + (1+2(1-\zeta))GZ_l^{-1}h^{-2\zeta} - (1+2(\zeta-1))GX_l h^{2(\zeta-2)}}$$

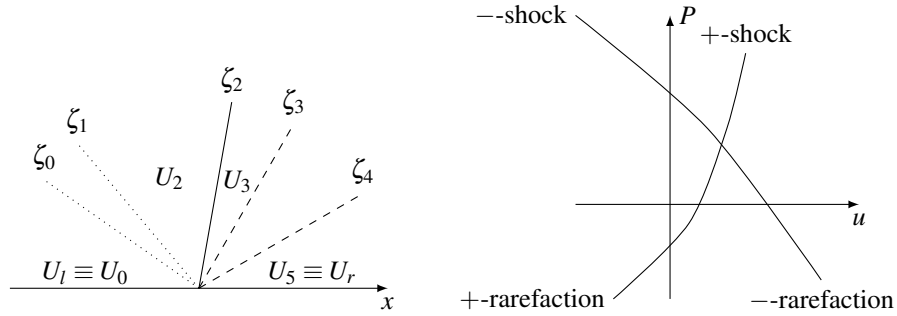
$$\xi \geq \lambda_l - \int_{h_l}^{h_1(\xi)} dh \frac{3gh + (4\zeta^2 - 14\zeta + 12)GZ_l^{-1}h^{2(1-\zeta)} + 2\zeta(1-2\zeta)GX_l h^{2(\zeta-1)}}{2h\sqrt{gh + (1+2(1-\zeta))GZ_l^{-1}h^{2(1-\zeta)} - (1+2(\zeta-1))GX_l h^{2(\zeta-1)}}}$$

when $h_2 \leftarrow h_1(\xi) \leq h_l$; a (h_3, u_3) -locus given by $u_3 = u_r + \sqrt{(h_r^{-1} - h_3^{-1})(P_3 - P_r)}$ on the other hand when $h_3 \geq h_r$ and, when $h_3 \leftarrow h_4(\xi) \leq h_r$,

$$u_3 \leftarrow u_r(\xi) = u_r + \int_{h_r}^{h_4(\xi)} dh \sqrt{gh^{-1} + (1+2(1-\zeta))GZ_r^{-1}h^{-2\zeta} - (1+2(\zeta-1))GX_r h^{2(\zeta-2)}}$$

$$\xi \leq \lambda_r + \int_{h_r}^{h_4(\xi)} dh \frac{3gh + (4\zeta^2 - 14\zeta + 12)GZ_r^{-1}h^{2(1-\zeta)} + 2\zeta(1-2\zeta)GX_r h^{2(\zeta-1)}}{2h\sqrt{gh + (1+2(1-\zeta))GZ_r^{-1}h^{2(1-\zeta)} - (1+2(\zeta-1))GX_r h^{2(\zeta-1)}}}.$$

Theorem 1. Given $\xi \in [0, \frac{1}{2}]$, $g > 0$, $G > 0$, the Riemann problem for gSV admits a unique admissible weak solution in \mathcal{U} for all $U_l, U_r \in \mathcal{U}$; this solution is piecewise continuous and differentiable with at most 5 discontinuity lines in $(x, t) \in \mathbb{R} \times \mathbb{R}^+$.



Proof. It suffices to show that there exists one unique solution satisfying (31) for all $U_l, U_r \in \mathcal{U}$. Now, it holds $\partial_h P > 0$ and one can use $(u, P, X, Z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_*^+ \times \mathbb{R}_*^+$ as parametrization of the state space \mathcal{U} (with $P \in \mathbb{R}_*^+$ when $\xi = \frac{1}{2}$), see figures above. Moreover, $\partial_P u = (\partial_h P)^{-1} \partial_h u$ is negative along the (h_2, u_2) -locus, strictly except at (h_l, u_l) , and positive along the (h_3, u_3) -locus, strictly except at (h_r, u_r) .

This is indeed easily established using $\partial_h u = (\partial_\zeta h)^{-1} \partial_\zeta u$ for rarefaction part; $\partial_h u = \pm \frac{P_* - P + h^2 \partial_h P (h^{-1} - h_*^{-1})}{2\sqrt{(h_*^{-1} - h^{-1})(P - P_*)}}$ for shock part, where $\partial_h P > 0$ and P is monotone increasing while h^{-1} is monotone decreasing thus $P \geq P_*$, $h^{-1} \leq h_*^{-1}$ when $h \geq h_*$ with $*$ = l/r .

So finally, since $(u_3|_{X_r, Z_r} - u_2|_{X_l, Z_l}) \rightarrow -\infty$ as $h = h_2 = h_3 \rightarrow 0^+$ and $(u_3|_{X_r, Z_r} - u_2|_{X_l, Z_l}) \rightarrow +\infty$ as $h = h_2 = h_3 \rightarrow +\infty$, there exists one, and only one, $P = P_2 = P_3$ zero of the continuous strictly non-decreasing function $(u_3|_{X_r, Z_r} - u_2|_{X_l, Z_l})$.

Note that it is not clear yet whether the unique Riemann solutions constructed above under Lax admissibility condition always satisfy the entropy dissipation (11). Classically, this is ensured for weak shocks only, using the asymptotic expansion of the *convex* entropy F as usual (see e.g. [7, Chap. VI]) like in Saint-Venant case $G = 0$ with small initial data. Interestingly, the latter limit case can be recovered in the limit $G \rightarrow 0^+$ also for small initial data only.

Corollary 1. *When $G \rightarrow 0^+$ one recovers the usual Riemann solution to the standard Saint-Venant system $G = 0$ (σ_{xx}, σ_{zz} then being “passive tracers”) only for initial data such that U_l, U_r are close enough within \mathcal{U} . In particular, it is not possible to reach piecewise continuous and differentiable Riemann solutions with a vacuum state $h = 0$ as the limits of bounded continuous sequences of Riemann solutions when $G > 0$, as opposed to the standard Saint-Venant case $G = 0$.*

Proof. The limit $h \rightarrow 0$ can only be reached through rarefaction waves. When $G = 0$, this necessarily occurs for large initial data. But when $G > 0$, the integrals defining the rarefaction waves are not well-defined (bounded) as $h_1 \rightarrow 0$ (−-field) or $h_4 \rightarrow 0$ (+-field), so this cannot occur for bounded (continuous sequences of) solutions.

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